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The compatibility equations and the pole to the Mohr circle

JONATHAN CUTLER and DAVID ELLIOTT*

Department of Earth and Planetary Science,
The Johns Hopkins University,
Baltimore, MD 21218, U.S.A.

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Abstract—We derive a new form of the compatibility equations for large deformations. These equations show that in a continuous inhomogeneous deformation, the strain gradients are related to the curvatures of the principal strain trajectories. In the case of uniform area strain, the equations express a direct relationship between the shape of the strain ellipse at a point and the curvatures of the principal trajectories. These relationships become particularly useful if the fabric in a deformed rock is taken as parallel to the principal strain trajectories.

We demonstrate that the compatibility equations provide important strain information for several geologically interesting special cases: uniform area strain, compaction of a bed, fanned or axial planar cleavage, and some three-dimensional structures such as cylindrical folds. We also show that in a three-dimensional structure with one straight strain trajectory there will always be uniform area strain in the cross-section normal to the straight trajectory, as long as the volume strain in the structure is uniform.

The pole of the finite strain Mohr circle is a unique point on the circle which graphically relates the state of strain in a body to its orientation in the physical plane. If a set of Mohr circles describes an inhomogeneous state of strain, then the curve connecting the poles of these circles is called the pole curve. We derive an exact analytical expression for the pole curve which applies to ductile deformation zones, refracted cleavage, and deformed stratigraphic sections; all with uniform area strain. For these special cases, the pole curve describes the entire strain field for the deformation as long as we know the strain at any one point in the deformed zone.

NOMENCLATURE

x_i the axes of a Cartesian coordinate system,
s_i the axes of an orthogonal curvilinear coordinate system everywhere tangent to the principal strain trajectories,
t_{ij} the reciprocal left stretch tensor,
λ'_{ij} the Cauchy strain tensor,
A the second invariant of the Cauchy tensor, also the square of the second invariant of the reciprocal left stretch tensor,
ψ the angular shear strain,
K_i the curvature of the principal strain trajectories,
θ' the angle between the tangent to s_1 and an arbitrary reference line,
ω the angle of rigid rotation,
B the constant value of λ'_{11} in a ductile deformation zone,
l_0 the original thickness of a small material element,
l_{22} the deformed thickness of a small material element,
L_0 the original thickness of a stratigraphic unit,
L_{22} the deformed thickness of a stratigraphic unit.

INTRODUCTION

UNFORTUNATELY, we never find strain markers at every point of interest in a deformed body of rock. What we need to unstrain a rock is an analytical expression for the whole strain field which can be deduced from a more ubiquitous property of the deformed rock, possibly cleavage. An analytical expression of this type is embodied in the compatibility equations for large deformations.

The strain in a continuously deformed rock cannot vary randomly from point to point. The common side of any two adjacent, deformed material elements must be the same length on both elements, or else discontinuities will develop. This idea can be extended to every side of each infinitely small deformed element in an entire body of rock. Thus, the simple stipulation of continuity, that neither gaps nor overlaps develop, puts constraints on how the strain can vary in the rock. The mathematical description of this stipulation is the compatibility relationship.

One reason that compatibility has received so little attention is that, in standard form, each of the two equations is a partial differential equation with two independent and four dependent variables. We will show two ways that the compatibility equations may be simplified substantially, revealing a significant amount of geological information. The methods which will be employed are (1) a coordinate transformation and (2) the consideration of special strain geometries.

THE EQUATIONS IN STANDARD FORM

Any very small material element in an inhomogeneous deformation field can be considered homogeneously deformed. According to the limit theorem, this approximation becomes exact for an infinitely small element. Since we cannot measure the strain in an infinitely small element, we must decide how large each material element can be without seriously violating the homogeneity approximation. The practical problem we face is to keep the size of the homogeously strained element small, relative to the scale of the inhomogeneous deformation of interest. It may be valid, for example, to take the

* Deceased 30 August 1982.

average strain in an outcrop as a homogeneous part of a regional strain field. The derivation and transformation of the compatability equations does not rely on actually choosing some finite element size with which to operate, but this choice is implicit whenever we apply the equations to deformed rocks.

The assumption that rock is a continuum with smoothly varying strains is fundamental to the compatibility equations. To the geologist, this means that no faults cut the region being studied. This restriction, however, is relative to the scale of observation. On a microscopic scale, for example, rocks may become a set of inhomogeneously strained crystals with discontinuities at the grain boundaries, violating the continuum assumption. Yet, these grain boundaries do not prevent us from applying the compatibility concept to deformations on the scale of an outcrop. In other words, discontinuities only become problematical as they approach the scale of the deformation being studied.

The application of the compatibility concept to large deformations is not new. In the continuum mechanics literature, for example, discussions of compatibility may be found in Gol'denblat (1962, pp. 74–82) and Truesdell & Toupin (1960, pp. 271–272), but this work is presented in a very abstract manner. The form of the compatibility equations stated by Gol'denblat, for instance, has over 65 terms in a single equation. No simplification or application of the compatibility equations is suggested in these works, leaving the reader with the impression that compatibility for large deformations is just a mathematical oddity without practical application.

There is some geological application of the much simpler compatibility equations for infinitesimal deformations. Schwerdtner (1976), for example, used a simple a-priori extension of the infinitesimal equations to demonstrate the importance of vertical as well as horizontal finite strain gradients in determining crustal shortening.

Cobbold (1977, 1980) produced the first full treatment of finite strain compatibility in the geological literature, and his equations are much simpler than any in the continuum mechanics literature. Cobbold's derivation forms the basis for our work, and we have presented a modified form of it in Appendix 1. This derivation shows that the two-dimensional compatibility equations for finite deformations in *standard form* are:

$$
\begin{aligned}
-A\,\frac{\partial\omega}{\partial x_1} &= \frac{1}{2}\,\frac{\partial\lambda'_{11}}{\partial x_2} - t_{11}\,\frac{\partial t_{12}}{\partial x_1} - t_{12}\,\frac{\partial t_{22}}{\partial x_1}\\
A\,\frac{\partial\omega}{\partial x_2} &= \frac{1}{2}\,\frac{\partial\lambda'_{22}}{\partial x_1} - t_{22}\,\frac{\partial t_{12}}{\partial x_2} - t_{12}\,\frac{\partial t_{11}}{\partial x_2};
\end{aligned}
\qquad (1)
$$

where x_1 and x_2 represent a coordinate system drawn on the deformed rock; λ'_{11} and λ'_{22} are the reciprocal quadratic elongations; t_{11}, t_{22} and t_{12} are the elements of the reciprocal left stretch tensor; A is the reciprocal quadratic area strain; and ω is the reciprocal rotation.

Certain steps in the derivation of the compatibility equations require the coordinate system we use to be orthogonal. Since we are always discussing the strain at a point, however, the orthogonality of the coordinate axes is only necessary at their point of intersection. The most general coordinate system in which we can use the compatibility equations is, therefore, an orthogonal curvilinear system, of which the Cartesian coordinate system is a special case.

Considerable simplification of the equations is immediately achieved by letting x_1 and x_2 be the special curvilinear coordinate frame whose axes are always parallel to the principal directions. In this special coordinate frame, all the off-diagonal components of the reciprocal pure strain tensor will be zero. To emphasize that we are using principal coordinates, x_1 and x_2 have been replaced by s_1 and s_2. The compatibility equations in the principal coordinate frame are:

$$
\begin{aligned}
-2A\,\frac{\partial\omega}{\partial s_1} &= \frac{\partial\lambda'_1}{\partial s_2}\\
2A\,\frac{\partial\omega}{\partial s_2} &= \frac{\partial\lambda'_2}{\partial s_1}.
\end{aligned}
\qquad (2)
$$

These equations are similar to those of Cobbold (1980, equation 5).

SIMPLIFICATION BY TRANSFORMATION

Our next simplification involves the application of a coordinate transformation, for which it is necessary that the old coordinate axes can be uniquely expressed as functions of the new ones. The strain field itself is invariant with respect to coordinate transformations, but our description of it may be simpler in one coordinate frame than in another. If we can find a special coordinate system with only one independent variable, then our description of the strain field will be in its simplest form (Appendix 2). In order to express the two-dimensional strain field with only one independent variable, we will have to put some restrictions on the path of integration in the physical plane.

Let s_1 and s_2 be axes of a principal curvilinear coordinate frame. Define the angle θ' between the tangent to the s_1 curve at some point and some arbitrary, but fixed, reference line (Fig. 1). Changes in the value of θ' are a function of changes in position along s_1. Because s_1 and s_2 are defined as perpendicular where they intersect, we could equally define θ' as the angle between a line perpendicular to s_2 and the same reference line. Thus, changes in θ' are also a function of changes in s_2. If we

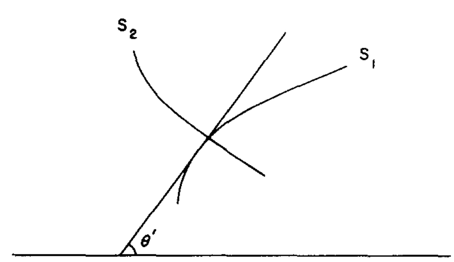

Fig. 1. The relationship of s_1 and s_2 to θ'.

now restrict ourselves to a monotonically increasing or decreasing segment of s_1 and s_2, then a change in position along that curve will be a single valued function of θ' along this segment. We have, therefore, fulfilled the necessary condition for a coordinate transformation, namely that the old coordinate system can be expressed as a unique function of the new one.

When we transform the compatibility equations from the s_1 and s_2 system to the θ' system, we find that many of the new terms represent a change in θ' with respect to a change in arc length along s_1 and s_2, which is the definition of the curvatures K_1 and K_2, respectively. Therefore,

$$\begin{aligned} -2A \frac{\partial \omega}{\partial \theta'} K_1 &= \frac{\partial \lambda_1'}{\partial \theta'} K_2 \\ 2A \frac{\partial \omega}{\partial \theta'} K_2 &= \frac{\partial \lambda_2'}{\partial \theta'} K_1. \end{aligned} \tag{3}$$

The only independent variable in these equations is θ', so all of the terms containing θ' in (3) can be considered as total differentials. Consequently, θ' can be algebraically eliminated from the equations, thus:

$$\begin{aligned} -2A\,\mathrm{d}\omega K_1 &= \mathrm{d}\lambda_1' K_2 \\ 2A\,\mathrm{d}\omega K_2 &= \mathrm{d}\lambda_2' K_1, \end{aligned} \tag{4}$$

which is the *reduced form* of the compatibility equations for large deformations. A somewhat more useful reduced form of these equations can be derived by combining the above equations:

$$-\frac{\mathrm{d}\lambda_1'}{\mathrm{d}\lambda_2'} = \left(\frac{K_1}{K_2}\right)^2 \tag{5}$$

and

$$(2A\,\mathrm{d}\omega)^2 = -\mathrm{d}\lambda_1'\mathrm{d}\lambda_2'. \tag{6}$$

We must still remember that position changes are restricted to monotonically increasing or decreasing segments of the s_1 and s_2 curves in the above equations. Our discussion will be concerned primarily with equation (5), since the rotational term in equation (6) is inherently difficult to measure.

EXPERIMENTAL VERIFICATION OF EQUATIONS

Before applying the compatibility equations to naturally deformed rocks, we need to verify that our mathematical manipulations do indeed have physical meaning. We test the compatibility equation (equation 5) in a simple experiment involving a deformation for which the strain gradients and the curvatures of the principal trajectories can be evaluated independently. We can then evaluate the left-hand side of equation (5) separately from the right-hand side. These independently determined quantities should be equal, which they are, and we therefore conclude that the reduced form of the compatibility equations is accurate.

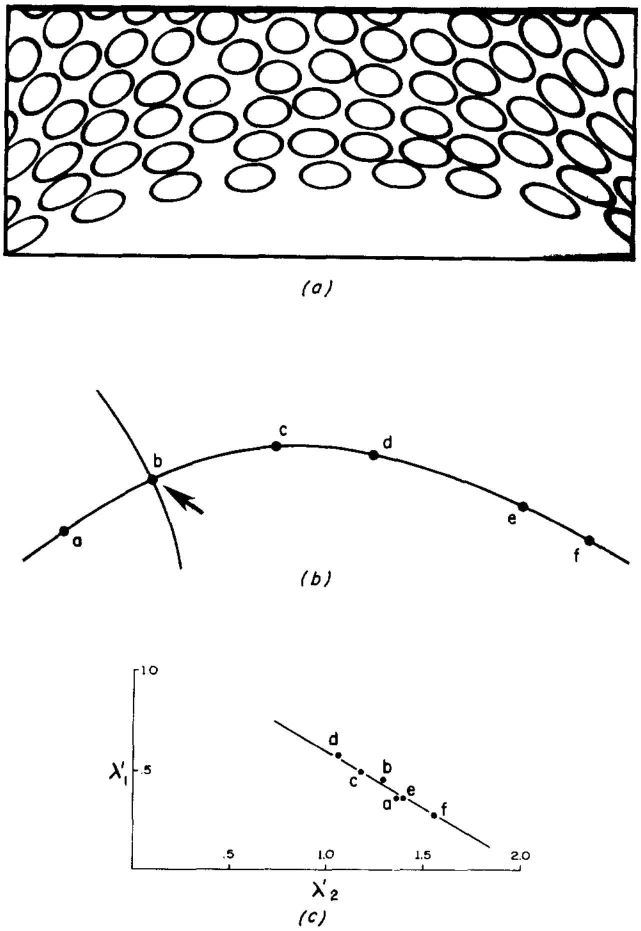

Fig. 2. Verifying the compatibility equations. (a) Photograph of inhomogeneously stretched rubber sheet. Ellipses were initially circles of equal area. (b) Principal strain trajectories traced from photograph. Strains were measured at points designated **a–f**. (c) Graphical technique used to measure strain gradients at point **b**.

The experiment delineated here simulates a two-dimensional deformation. Regardless of the material being deformed, application of the two-dimensional equation implies that the planar section being studied is a principal plane of the deformation. This experiment is independent of the scale and type of material chosen for the deformation, as long as the deformation is continuous. We performed the experiment using a sheet of natural rubber measuring approximately 7.5 × 20 × 0.05 centimeters.

Method

Draw or otherwise make a large number of closely spaced circles on the rubber sheet. The circles should be small enough to deform homogeneously during an inhomogeneous deformation of the rubber sheet. Photograph the rubber sheet in its unstretched state. Now stretch the rubber sheet such that the resulting deformation is obviously inhomogeneous. The small circles should still deform to ellipses, even though the strain changes from ellipse to ellipse. Photograph the deformed rubber sheet (Fig. 2a). If both the undeformed and deformed states are photographed at the same scale, then the photographs can be enlarged or reduced to any convenient size in order to measure the strains and curvatures.

Working from the photograph of the deformed rubber sheet, trace off any pair of principal trajectories. One trajectory should be everywhere parallel to the long axes of the ellipses and the other should be everywhere parallel to the short axes. The point at which these trajectories intersect is the point for which the compatibility equations will be tested. The first step is to measure the curvature of each of these principal trajectories at their point of intersection, a task which we undertook using a digitizer and a small computer. Use the curvatures of the principal trajectories to evaluate the right-hand side of equation (5).

To evaluate the left-hand side of equation (5), we need to measure the strain gradients at the same point for which we measured the curvatures. From the derivation of the compatibility equations we know that the strain gradients must be evaluated along one of the principal trajectories. Choose several points, along either of the trajectories, for which the strain can be easily measured (Fig. 2b). One of these points must be the point at which the principal trajectories intersect. Now measure λ_1' and λ_2' for each of the selected points and plot the strains for each point on a graph of λ_1' vs λ_2'. Fit a curve to the plotted points. For natural rubber, this curve turns out to be a straight line (Fig. 2c), which may not be the case with materials which deform plastically. The fact that the strain gradients are linear does not affect the experiment.

If the fitted curve is a straight line, then the slope of this line represents the left-hand side of equation (5). If the fitted curve is not a straight line, then draw a line which is tangent to the curve at the strain point which corresponds to the material point where the trajectories intersect. The slope of this tangent line is equal to the ratio of the strain gradients at the point of intersection of the trajectories. To evaluate the left-hand side of equation (5), take the negative of this slope. Compare the values obtained for the two sides of equation (5).

Results

In our experiment, the difference between the ratio of the strain gradients (0.57) and the square of the ratio of the curvatures (0.52) is less than 10%, which we consider to be within the limits of experimental error. We conclude that equation (5) is valid and has physical meaning.

PURE FLATTENING AND COMPACTION OF A BED

Consider bed surfaces that were straight and parallel at the time of deposition, and which retain this configuration after deformation. Let flattening occur such that the strain trajectories are everywhere oriented parallel and perpendicular to the bed. We call this type of deformation pure flattening. Geologically important cases of

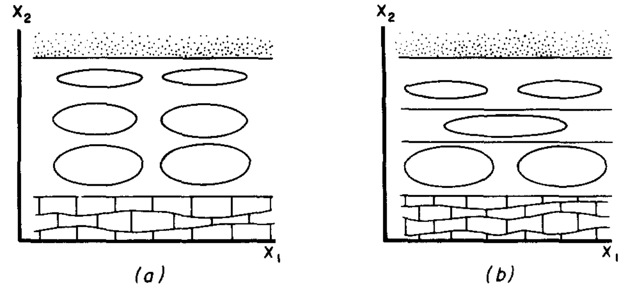

Fig. 3. Only three styles of deformation satisfy the compatibility equations in the case of a pure flattening: (a) Area strain varying across the bed only. (b) Inhomogeneous pure shear, with the discontinuities being required by compatibility, and homogeneous pure shear over the whole layer, which is not depicted.

pure flattening include zones of locally enhanced pressure solution and beds which have been compacted under an overlying sediment load.

It is advantageous to choose a Cartesian coordinate system parallel to the principal directions. From the relationships between the elements of the Cauchy strain tensor and the reciprocal left stretch tensor (Appendix 1, equations 7 and 8), we can see that $t_{12} = 0$ for pure flattening in the principal coordinate frame. Because the surfaces of the flattened bed are parallel both before and after the deformation, we also know that the rotational gradient is zero. The standard form of the compatibility equations (1) thus becomes:

$$\frac{\partial \lambda_1'}{\partial x_2} = 0 \qquad \frac{\partial \lambda_2'}{\partial x_1} = 0. \tag{7}$$

The parallel nature of the bedding surface also permits us to say:

$$\frac{\partial \lambda_1'}{\partial x_1} = 0. \tag{8}$$

Thus, the only non-zero strain gradient in the compacted bed case is $\partial\lambda_2'/\partial x_2$.

There are two different ways to satisfy these equations. Either (1) $\partial\lambda_2'/\partial x_2 = 0$, and the style of deformation is homogeneous pure shear, or (2) $\partial\lambda_2'/\partial x_2 \neq 0$, and there is an area strain gradient across the bed which must take place in such a way that only λ_2' changes (Fig. 3a). The alternative is that there are discontinuities in the material and the equations are not valid (Fig. 3b). Discontinuities which could cause the compatibility relationship to breakdown might take the form of bedding surfaces with slip, stylolites, or fractures. In the event that the strain is discontinuous, we might also expect to see λ_1' varying across the beds. All of the strains would still be constant parallel to the bedding.

UNIFORM AREA STRAIN

In a later section we shall show that deformations with uniform area strain may be fairly common. For now we shall assume that it is possible to determine independently that the area strain is uniform over the deformed region in which we are interested. One measure of the area strain is the second invariant of the Cauchy strain tensor:

$$A = \lambda_1' \lambda_2'. \tag{9}$$

When $A = 1$, there is no area strain; when $0 < A < 1$, the deformed area is larger than the undeformed area; and when $1 < A <$ infinity, the deformed area is smaller than the undeformed area. Putting equation (9) into differential form,

$$\mathrm{d}A = \lambda_1' \mathrm{d}\lambda_2' + \mathrm{d}\lambda_1' \lambda_2'. \tag{10}$$

Now, if we permit area strain, but require that the area strain remain uniform over the region of interest, then $\mathrm{d}A = 0$ and equation (10) becomes:

$$-\frac{\mathrm{d}\lambda_1'}{\mathrm{d}\lambda_2'} = \frac{\lambda_1'}{\lambda_2'}. \tag{11}$$

This equation has a term in common with the reduced form of the compatibility equation in principal coordinates (5). Eliminating the common terms between these two equations:

$$\frac{\lambda_1'}{\lambda_2'} = \left(\frac{K_1}{K_2}\right)^2, \tag{12}$$

which states that the axial ratio of the strain ellipse at a point is exactly determined by the ratio of the curvatures of the principal strain trajectories for the case of uniform area strain. Simplifying (12):

$$\textit{Axial Ratio} = K_2/K_1. \tag{13}$$

From equation (13) it is clear that if $K_2 = 0$, then the curvature of the λ_1' trajectory must also be equal to zero. The only other way for K_2 to equal zero is if λ_1' equals zero, which violates conservation of mass.

The geological implication of this situation is significant, especially in the case of fanned cleavage. If we observe a set of non-parallel, but straight strain trajectories, then there must either be a variable area strain or discontinuities in the rock. These discontinuities could take the form of bedding plane slip, faults, large stylolites, or veins.

We emphasize that a uniform area strain is not the same as no area strain. In the next section we will show that uniform area strain may be a case of rather wide applicability.

THREE-DIMENSIONAL STRAIN DISTRIBUTIONS

One straight strain trajectory

Consider a three-dimensional deformation where one of the three principal directions, possibly the stretching direction or the cleavage, is a straight line in the rock. The reduced form of the compatibility equations (5) can be written for each principal surface in the three-dimensional deformation. The fact that the principal planes may be curved surfaces does not prevent this generalization, since s_1, s_2 and s_3 will still represent an orthogonal curvilinear coordinate system. This set of equations will take the form:

$$-\frac{\mathrm{d}\lambda_1'}{\mathrm{d}\lambda_2'} = \left(\frac{K_1}{K_2}\right)^2 \tag{14}$$

$$-\frac{\mathrm{d}\lambda_2'}{\mathrm{d}\lambda_3'} = \left(\frac{K_2}{K_3}\right)^2 \tag{15}$$

$$-\frac{\mathrm{d}\lambda_1'}{\mathrm{d}\lambda_3'} = \left(\frac{K_1}{K_3}\right)^2. \tag{16}$$

If one of the principal strain trajectories is a straight line, the curvature of that trajectory will be zero. Often $K_2 = 0$, but $K_1 \neq 0$ and $K_3 \neq 0$, as in the case of a cylindrical fold. Then, from equation (15) we see immediately that $\mathrm{d}\lambda_2' = 0$, so that λ_2' must be constant in the $\lambda_2'\lambda_3'$ surface. Also, from equation (14) we can see that λ_2' must also be constant in the $\lambda_1'\lambda_2'$ surface. We can conclude that λ_2' is constant in any three-dimensional structure with a straight intermediate strain trajectory. The same argument is equally valid for any three-dimensional deformation with a straight strain trajectory; the strain parallel to this trajectory always being constant.

Two straight strain trajectories

Now consider a region where both the cleavage and the stretching direction plot as point maxima on a stereonet, so that two of the three principal strain trajectories are straight. In other words, $K_2 = 0$ and $K_3 = 0$, and both λ_2' and λ_3' are constant in the deformed structure. This could be the case on a local scale, such as along the planar axial surface of a cylindrical fold. Cobbold (1980, p. 382) uses the example of the bending of a beam to illustrate this situation.

One straight strain trajectory and uniform volume strain

One measure of volume strain in three dimensions is:

$$V = \lambda_1'\lambda_2'\lambda_3'. \tag{17}$$

If the volume strain is uniform, and if $K_2 = 0$, then λ_2' is a constant and equation (17) can be written:

$$\frac{V}{\lambda_2'} = \lambda_1'\lambda_3' = \text{constant}, \tag{18}$$

which says that the area strain in the $\lambda_1'\lambda_3'$ plane is uniform.

This discussion demonstrates that any cross-section taken normal to a straight strain trajectory will have uniform area strain, as long as the volume strain is also uniform. For example, if a ductile deformation zone has a stretching direction which is everywhere normal to the plane of the section, and if the volume strain is uniform, then the area strain in the plane of the section will also be uniform. Thus, the uniform area strain assumption is easily tested from the symmetry of the fabric, and we think that this case may prove to be fairly common. The uniform volume strain assumption might be testable by taking a series of density measurements and showing that the density remains constant over the region of interest.

THE POLE CURVE

A Mohr circle describes the two-dimensional state of strain in a homogeneously deformed rock. Direction on the Mohr diagram always means 'relative to the principal directions', and there is no reference to an external coordinate frame. If we want to record the physical orientation of our deformed material on the Mohr diagram, then we must use a point on the Mohr circle called the *pole*.

To locate the pole, draw the deformed objects on a piece of paper. On the same piece of paper construct the Mohr diagram for these objects. Next, take a material line, *l*, for which the strain is known and plot its strain point on the Mohr circle (Fig. 4). Now construct the line *l'* such that it goes through the strain point and is physically parallel to *l*. The other point in which *l'* intersects the Mohr circle is the pole.

The above procedure can be applied to any material line in the physical plane, and the position of the pole will be the same in every case. The formal definition of the pole is *the point of intersection, on the Mohr circle, of all lines which go through strain points and are parallel to the material lines which those points represent.*

The pole of the Mohr circle is widely used for describing states of stress (e.g. Ford & Alexander 1977, p. 70, Mandl & Shippam 1981, p. 96). In fact, the pole can be defined for a Mohr circle representation of any second rank tensor, including the reciprocal left stretch tensor and Cauchy strain tensor. Note that with the Mohr circle for stress, the pole is drawn parallel to the normal of a plane on which the stress is known. For finite strain, on the other hand, we define the pole as parallel to the line whose strain is known.

At The Johns Hopkins University we have been solving finite strain problems using the pole construction since 1969, and we think that it can be used to considerable advantage. For example, to find the orientation of the material line corresponding to a particular strain point on the Mohr circle, simply draw a line from the strain point to the pole. This line is parallel to the material line in the deformed rock which has the strain represented by that point. This is an easy way to find the principal directions in the deformed rock if the Mohr circle is known (line k, Fig. 4).

After studying Prager's elegant graphical construction using the pole for the stress Mohr circle to deduce slip line fields for a flowing, perfectly plastic material (see Ford & Alexander 1977, pp. 491–513), we thought that a somewhat similar construction might be valid for inhomogeneous finite strains. Consider a deformed planar cross-section and draw an arbitrary line, m, on this section (Fig. 5). Choose several points along m and construct the Mohr circle for strain at each of these points, drawing all of the Mohr circles on the same strain plane. Each of these Mohr circles will have a pole, and the curved line, m', which goes through all of these poles will represent the state of strain along m. In other words, line m on the physical plane has an image m' in the strain plane, and this image is called the pole curve. Note that when we choose a Cartesian coordinate frame for the physical plane it should be oriented such that the x_1 direction is parallel to the horizontal axis of the strain plane.

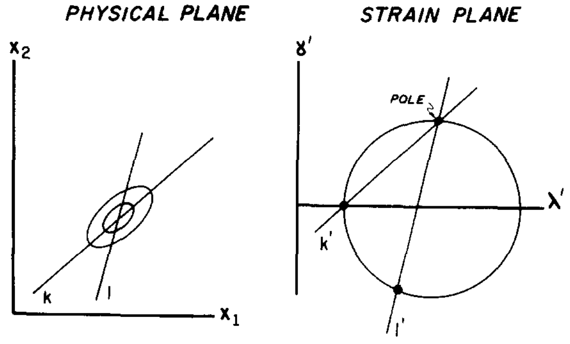

Fig. 4. The pole to the Mohr circle. (See text for explanation).

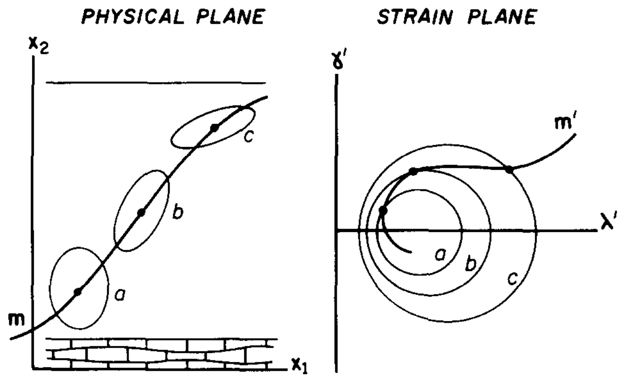

Fig. 5. Construction of the pole curve. The variation in strain along line m in the physical plane is represented by the pole curve, m', in the strain plane.

In order to find an analytical expression for the pole curve, we must first determine which variables describe the position of the pole in the strain plane. A vertical line in the physical plane will have the strain λ'_{22}, since we take this to be the x_2 direction. Drawing a vertical line through the pole of a Mohr circle, we can see that all poles have λ'_{22} as one of their coordinates (Fig. 6). The shear on the line parallel to x_2 is λ'_{12} by definition, so λ'_{12} will be the other coordinate of the pole in Mohr space. Any analytical expression for the pole curve, therefore, must describe how λ'_{22} varies as a function of λ'_{12}. In the next section, we will use the compatibility equations along with a geologically important strain geometry to arrive at an expression for the pole curve.

In summary, it is possible to represent the complete state of strain along a given material line with a large number of adjacent, homogeneously deformed material

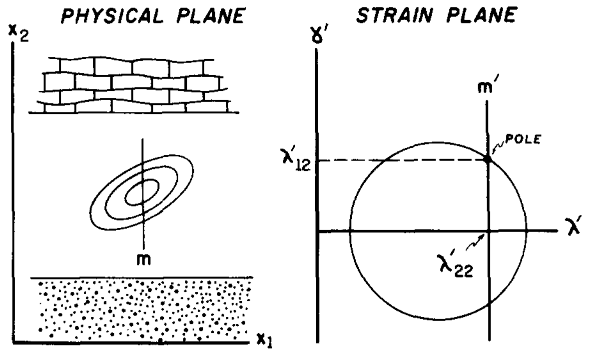

Fig. 6. The pole for any homogeneous deformation has the coordinates (λ'_{22}, λ'_{12}). Every point on the pole curve represents the pole to one Mohr circle, and therefore has these coordinates.

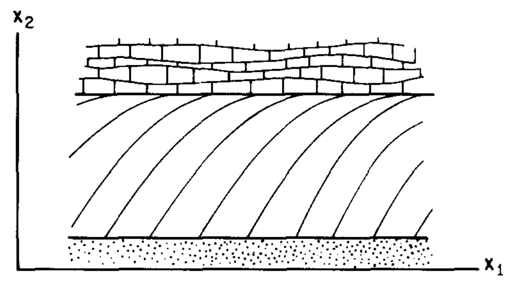

Fig. 7. Constant strain in one direction. Here the parallel strain trajectories and parallel bounding surfaces indicate that the strain is constant in the x_1 direction. This could be a ductile deformation zone, refracted cleavage, or a deformed stratigraphic unit.

elements. Each deformed element will have a Mohr circle, and each Mohr circle will have a pole. If all of these Mohr circles are plotted on the same Mohr plane, then all of their poles can be connected into a continuous and unique curve, called the pole curve. Every material line in the physical plane thus maps onto a unique line, or pole curve, in the strain plane.

Ductile deformation zones and refracted cleavage

Draw a straight line on a deformed cross-section. If all of the principal strain trajectories crossing that line are parallel to each other, and if both boundaries of the deformed zone are parallel to the constructed line, then the strain is constant along this line. Geologic examples of this important special case are refracted cleavage, ductile deformation zones, and some deformed stratigraphic sections.

It is advantageous to choose a Cartesian coordinate system with x_1 parallel to the direction of constant strain and the boundaries of the deformed zone (Fig. 7). If all of the strains are constant in the x_1 direction, then all of the strain gradients in this direction must be zero. The standard form of the finite strain compatibility equations (1) becomes:

$$\frac{\partial \lambda'_{11}}{\partial x_2} = 0$$
$$-A\,\frac{\partial \omega}{\partial x_2} = t_{22}\,\frac{\partial t_{12}}{\partial x_2} + t_{12}\,\frac{\partial t_{11}}{\partial x_2}. \qquad (19)$$

The compatibility equations now contain only one independent variable, x_2, and the partial differentials can be converted to total differentials. Multiplying both equations through by dx_2 yields:

$$d\lambda'_{11} = 0, \quad \text{or} \quad \lambda'_{11} = \text{constant}$$
$$\text{and} \qquad -A\,d\omega = t_{22}dt_{12} = t_{12}dt_{11}. \qquad (20)$$

In other words, the compatibility equations for this strain geometry show that λ'_{11} has the same value everywhere in the deformed cross-section.

We have already used the second variant of the Cauchy strain tensor as a measure of area strain (equation 9). In a non-principal frame, this area strain is:

$$A = \lambda'_{11}\lambda'_{22} - (\lambda'_{12})^2. \qquad (21)$$

If the area strain does not change from point to point within our deformed zone, then both A and λ'_{11} will be constants in the above equation. Rewriting equation (21) to emphasize this fact:

$$(\lambda'_{12})^2 = B\left(\lambda'_{22} - \frac{A}{B}\right), \qquad (22)$$

where $B = \lambda'_{11}$, and both A and B are constants.

Equation (22) gives λ'_{12} as a function of λ'_{22}, and is therefore the equation of the pole curve for the deformation. This one pole curve is valid for all material lines in the deformed zone, since we have expressed the strain field as a function of only one position variable, x_2. If we know the strain at any one point in the deformed zone, then we can evaluate the constants A and B and draw the pole curve. Equation (22) is the equation of a parabola, so we expect the pole curve for any equal area ductile deformation zone to be parabolic in form.

The parabola always closes to the left and its vertex is on the horizontal axis, having coordinates ($\lambda'_{12} = 0$, $\lambda'_{22} = A/B$). If the strain in the deformed zone reaches the vertex of the pole curve, then at this point our Cartesian coordinate frame coincides with the principal coordinate frame. Thus, at the vertex of a parabolic pole curve we have the degenerate case of pure flattening, or a ductile deformation zone with no shear.

Now look at the case of a ductile deformation zone which is only undergoing simple shear, so that there is neither area strain nor longitudinal strain parallel to the zone boundaries. In other words, $A = 1$ and $B = \lambda'_{11} = 1$, and equation (22) can be re-written as:

$$(\lambda'_{12})^2 = \lambda'_{22} - 1, \qquad (23)$$

and the pole curve is a parabola whose vertex is at $\lambda'_{22} = 1$. Thus, the vertex represents a point in the deformed zone at which there is no strain at all; a situation often assumed to exist at the margin of ductile deformation zones (Ramsay & Graham 1970, p. 799).

We have seen that for the special case of the ductile deformation zone and refracted cleavage, all possible strain states in the deformed rock plot onto a parabolic

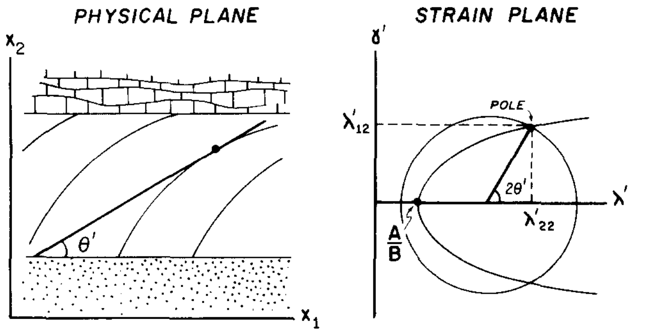

Fig. 8. The pole curve for a deformation having constant strain in one direction is a parabola. For any point on the pole curve, a Mohr circle can be drawn which represents the state of strain at one point on a given principal trajectory in the physical plane. Because the strain is constant parallel to x_1, this Mohr circle actually represents the state of strain everywhere on a line which is parallel to x_1 and goes through the point for which θ' is measured.

pole curve in the strain plane. Now, every point on the pole curve represents one Mohr circle and this Mohr circle represents the state of strain everywhere on a particular material line which is parallel to x_1, since we have required the strain to be constant in that direction. If we know the principal directions for a point on the pole curve, then we can draw its Mohr circle and find the material line which this Mohr circle represents. To find the principal directions associated with any point on the pole curve, we use the Mohr circle identity:

$$\tan(2\theta') = \frac{2\lambda'_{12}}{\lambda'_{22} - \lambda'_{11}}, \text{ or } \theta' = \frac{1}{2}\tan^{-1}\left(\frac{2\lambda'_{12}}{\lambda'_{22} - B}\right), \tag{24}$$

where θ' is the angle that the principal direction will make with the x_1 axis in the physical plane. As long as there is a single valued relationship between θ' and the orientation of the principal strain trajectory, every θ' will define a unique point on any given strain trajectory in the physical plane (Fig. 8). It does not matter which strain trajectory we use, since all of the strain trajectories are assumed to be parallel along the ductile deformation zone. Thus, equation (22) permits us to take any point on the pole curve, draw its Mohr circle, and find its corresponding point in the physical plane. Once a pole curve is established, it is a good practice to mark values of θ' at regular intervals along the curve.

Since neglecting any component of the deformation can lead to large errors, we are generally wary of the simple shear assumption. We therefore prefer the less euphemistic term 'ductile deformation zone' to the term 'shear zone'. When constructing the pole curve for a ductile deformation zone, it may nonetheless be instructive to plot the simple shear parabola for comparison purposes.

Initial thickness of a deformed stratigraphic unit

In order to understand the internal parts of thrust belts, where large penetrative deformations are the rule, it is essential to develop methods by which we can restore deformed stratigraphic units to their original thickness. Cloos (1947, pp. 906–912) was the first to attempt to restore a stratigraphic section using strain measurements. Employing a single strain measurement, Cloos suggested that the current stratigraphic section in the hinge of the South Mountain fold, Maryland, is now about twice its original thickness. Cloos performed this calculation for only one point in the fold, however, neglecting possible inhomogeneities in the strain.

The numerical integration of strains was introduced by Ramsay (1969, pp. 58–62) as a way of restoring inhomogeneously strained stratigraphic sections. Ramsay's concept of strain integration is fundamental to the study of any inhomogeneous deformation, but his method for unstraining stratigraphic sections ignores the rotational component of the strain.

Hossack (1978) applied a modified form of the strain integration technique to the Bygdin conglomerate, Norway, using deformed pebbles as strain markers. To calculate the original thickness of a deformed section using Hossack's method, however, we must assume that the strain trajectories have not been rotated and that there is no area strain. While these assumptions are shown to be reasonable for the Bygdin conglomerate, they are too restrictive to have wide application.

We propose that many deformed stratigraphic sections have parallel boundaries and fit the criteria for constant strain in one direction, thus having a strain geometry similar to the ductile deformation zone. Furthermore, if the fabric suggests that the area strain is uniform (see our discussion of three-dimensional strain distributions), and if we know the state of strain at one point then we can immediately construct the pole curve for this deformation.

Consider a small, homogeneously deformed material element in a stratigraphic section. The material line which was originally perpendicular to bedding has now been sheared through an angle ψ, where:

$$\tan\psi = \frac{\lambda'_{12}}{\lambda'_{11}}, \text{ or } \psi = \tan^{-1}\left(\frac{\lambda'_{12}}{B}\right). \tag{25}$$

This material line (Fig. 9a) will have a deformed length of:

$$l_\psi = \frac{l_{22}}{\cos(\psi)}, \tag{26}$$

where l_{22} is the deformed thickness of the material element measured perpendicular to the bedding (Fig. 9a). From the definition of the reciprocal quadratic elongation we can write:

$$l_0 = l_\psi\sqrt{\lambda'_\psi}, \tag{27}$$

where l_0 is the original thickness of our small element. Now substituting (26) into (27) gives:

$$l_0 = \frac{\sqrt{\lambda'_\psi}}{\cos(\psi)}\, l_{22}. \tag{28}$$

Equation (28) is for one small material element in our deformed section. To find the total original thickness of

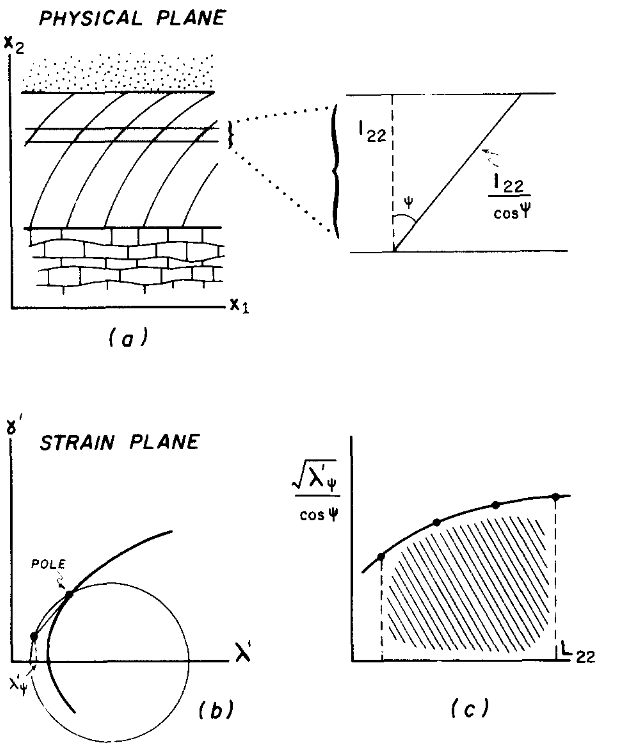

Fig. 9. Restoring a deformed stratigraphic section using the pole curve. (a) Cleavage or strain markers suggest that the strain is constant in one direction. Simple trigonometric relationships are established for a small material element in the deformed section. (b) If the strain is known at any one point in the section, the pole curve is easily constructed and λ'_ψ can be evaluated at different points in the section. (c) To numerically integrate equation (29), find the area under the $\sqrt{\lambda'_\psi}/\cos\psi$ vs L_{22} curve.

our deformed section, we write equation (28) in terms of the infinitesimal lengths and then integrate. The result is:

$$L_0 = \int_a^b \frac{\sqrt{\lambda'_\psi}}{\cos(\psi)}\, \mathrm{d}l_{22}, \tag{29}$$

where L_0 is the total original thickness of the unit. Equation (29) is conceptually similar to the relationships suggested by Hossack (1978, equation 5) and Ramsay (1969, p. 62).

In order to numerically integrate equation (29) we must know the value of $\sqrt{\lambda'_\psi}/\cos\psi$ at a number of discrete points in the deformed section, a task which is easily accomplished using the pole curve. Take a point in the deformed section and, using the orientation of the principal direction at that point, locate its corresponding point on the pole curve. The coordinates of this point in Mohr space allow us to calculate ψ from equation (25). We can then use either the pole or conventional Mohr circle methods to find the reciprocal quadratic elongation, λ'_ψ (Fig. 9b). We can now plot $\sqrt{\lambda'_\psi}/\cos\psi$ vs L_{22}, and the area under this curve will be the original thickness of our deformed section (Fig. 9c). Thus, if a parallel-sided stratigraphic section has uniform area strain and constant strain in one direction, we can use the pole curve and numerical integration to determine its original thickness. To do this we only need to know the state of strain at one point and the shape of the principal strain trajectories in the deformed rock.

Generalizing the pole curve

We have shown that for some special cases, a single pole curve can represent the state of strain everywhere in a deformed cross-section. In these cases, one point on the pole curve represents the state of strain on an entire material line in the physical plane. This simplification results from our requirement that the strain is a function of x_1 only, and is constant in the x_2 direction.

In the general case, however, the strain will vary with both x_1 and x_2. Thus the pole curve will only represent the state of strain along one material line, with each point on the pole curve referring to the state of strain at one material point on this line.

If we knew the general equation of the pole curve, it would be useful to plot the pole curves for all of the principal trajectories on the same strain plane. The equation for the pole curve would then be acting as a transformation law; converting the physical geometry of the strain into its corresponding strain field in the strain plane. We propose that this transformation law may be represented by the compatibility equations.

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APPENDIX 1: DERIVATION OF THE FINITE STRAIN COMPATIBILITY EQUATIONS

This derivation is essentially Cobbold's (1977), though we present it in a somewhat expanded form. We use matrix algebra and the matrix representation of tensors wherever possible.

In geology, we always observe and measure rocks in the deformed state, and we are therefore interested in the transformation which carries the deformed rock to its undeformed state. We will call this transformation a reciprocal deformation, and express it algebraically as:

$$\begin{aligned} X_1 &= f(x_1, x_2) \\ X_2 &= g(x_1, x_2), \end{aligned} \tag{A1}$$

where the small letters represent the deformed state and the capital letters the undeformed state. Rewriting equations (A1) in differential form:

$$\begin{aligned} \mathrm{d}X_1 &= \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\,\mathrm{d}x_2 \\ \mathrm{d}X_2 &= \frac{\partial g}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial g}{\partial x_2}\,\mathrm{d}x_2, \end{aligned} \tag{A2}$$

or in matrix form:

$$\mathbf{d} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix}, \tag{A3}$$

where $\mathbf{d}$ is the reciprocal deformation matrix, which transforms the strain ellipse back into a circle.

The reciprocal deformation matrix contains both rotational and pure strain information mixed together in a complex way, all described with respect to a coordinate system drawn on the deformed rock. We want to express all of the solid body rotation with one matrix, $\mathbf{r}$, and all of the changes in size and shape with another matrix **t**. This decomposition is based on Cauchy's fundamental theorem, which states that any deformation at a point can be decomposed into a translation, a rigid rotation, and a pure strain (Ericksen 1960, pp. 840–842, Truesdell & Toupin 1960, p. 274).

Because our strains are to be measured in a coordinate system drawn on the deformed rock, we must remove the shape change before applying the reciprocal rotation. In matrix algebra, successive deformation events accumulate as matrices toward the left (Elliott 1972, pp. 2622–2623, Truesdell & Toupin 1960, p. 246), so we have:

$$\mathbf{d} = \mathbf{rt} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}. \quad \text{(A4)}$$

The symmetric matrix **t** is called the reciprocal left stretch matrix. This nomenclature refers to the forward deformation which is the inverse of equation (A4):

$$\mathbf{D} = \mathbf{T}\,\mathbf{R}. \quad \text{(A5)}$$

Since we have used left polar decomposition for the forward deformation (Elliott 1970, pp. 2234–2235), **T** is called the left stretch matrix. Thus, we call the inverse of **T** the reciprocal left stretch matrix and represent it as **t**. Carrying out the multiplication indicated in equation (A4):

$$\mathbf{d} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} t_{11}\cos\omega - t_{12}\sin\omega & t_{12}\cos\omega - t_{22}\sin\omega \\ t_{11}\sin\omega + t_{12}\cos\omega & t_{12}\sin\omega + t_{22}\cos\omega \end{bmatrix}. \quad \text{(A6)}$$

Due to the utility of the Cauchy strain matrix in structural geology, we note the relationship between the Cauchy matrix λ' and the reciprocal left stretch matrix **t**:

$$\begin{bmatrix} \lambda'_{11} & \lambda'_{12} \\ \lambda'_{12} & \lambda'_{22} \end{bmatrix} = \begin{bmatrix} t_{11}^2 + t_{12}^2 & t_{12}(t_{11} + t_{12}) \\ t_{12}(t_{11} + t_{22}) & t_{22}^2 + t_{12}^2 \end{bmatrix}. \quad \text{(A7)}$$

And in the principal coordinate frame:

$$\lambda'_1 = t_1^2 \qquad \lambda'_2 = t_2^2. \quad \text{(A8)}$$

If the material remains continuous, then the deformation must vary in a smooth manner, which means that f' and g' must be continuous and differentiable functions. From calculus we know that the second order, mixed partial derivatives of a continuous function are equal, regardless of the order in which the derivatives are taken (see Thomas & Finney 1979, pp. 629–632 for a proof). Therefore

$$\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right), \quad \frac{\partial}{\partial x_2}\left(\frac{\partial g}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial g}{\partial x_2}\right). \quad \text{(A9)}$$

Substituting the elements of the reciprocal deformation matrix, from equation (A6) into equation (A9) we find:

$$\begin{aligned} \frac{\partial}{\partial x_2}[t_{11}\cos\omega - t_{12}\sin\omega] &= \frac{\partial}{\partial x_1}[t_{12}\cos\omega - t_{22}\sin\omega] \\ \frac{\partial}{\partial x_2}[t_{11}\sin\omega + t_{12}\cos\omega] &= \frac{\partial}{\partial x_1}[t_{12}\sin\omega + t_{22}\cos\omega], \end{aligned} \quad \text{(A10)}$$

which is the compatibility requirement for finite deformation. After expanding the derivates in equation (A10) and undertaking a considerable amount of algebra, we find that the trigonometric terms cancel and we are left with the compatibility equations in *standard form*:

$$\begin{aligned} -A\frac{\partial\omega}{\partial x_1} &= \frac{1}{2}\frac{\partial\lambda'_{11}}{\partial x_2} - t_{11}\frac{\partial t_{12}}{\partial x_1} - t_{12}\frac{\partial t_{22}}{\partial x_1} \\ A\frac{\partial\omega}{\partial x_2} &= \frac{1}{2}\frac{\partial\lambda'_{22}}{\partial x_1} - t_{22}\frac{\partial t_{12}}{\partial x_2} - t_{12}\frac{\partial t_{11}}{\partial x_2}, \end{aligned} \quad \text{(A11)}$$

where "A" is the reciprocal quadratic area strain, $A = (t_{11}t_{22} - t_{12}^2)^2$, and λ'_{11} and λ'_{22} have been substituted according to the relationships in equation (A7).

Equation (A11) differs from Cobbold (1977, equation 7) in two small ways. First, we have explicitly used the fact that **t** is symmetric, so that $t_{12} = t_{21}$. Second, our equations differ by a factor of -1, because in a right-handed coordinate frame, a counter-clockwise reciprocal rotation is positive. This sense of rotation is opposite to that implied by Cobbold (1977, equation 3), but in agreement with his later work (1980, equation 3).

APPENDIX 2: TRANSFORMATION OF THE COMPATIBILITY EQUATIONS

Define R_i to be a general, three-dimensional curvilinear coordinate system. Now define a new coordinate system, S_i, such that every point in the R_i system has a new and unique set of coordinates in the S_i system. The equations of the axial surfaces of the new coordinate system can be written in terms of the old coordinates as follows:

$$S_1(R_1,R_2,R_3) = b_1 \quad S_2(R_1,R_2,R_3) = b_2 \quad S_3(R_1,R_2,R_3) = b_3, \quad \text{(A12)}$$

and since the transformation is unique, we can also write:

$$R_1(S_1,S_2,S_3) = b'_1 \quad R_2(S_1,S_2,S_3) = b'_2 \quad R_3(S_1,S_2,S_3) = b'_3, \quad \text{(A13)}$$

where the b_n's are constants representing displacement of the origin of the S_i system relative to the origin of the R_i system. The uniqueness requirement is equivalent to saying that the coordinate axes of both coordinate systems intersect in only one point (for good discussions of coordinate systems see McConnell 1957, pp. 130–132, Gol'denblat 1962, pp. 25–27, and Morse & Feshbach 1953, pp. 21–25).

Now consider a general function F, where: $F = f(R_1,R_2,R_3)$. The physical meaning of $\partial F/\partial R_1$ is the amount that the value of F changes due to the R_1 component of any three-dimensional position change. In the R_i frame, we find the R_1 component of the position change by projecting the general position change onto the R_1 axis. We then evaluate how much the value of F changed for this projected displacement along R_1.

The physical meaning of this $\partial F/\partial R_1$ should not change just because we move into the S_i coordinate frame. By equation (A13) we can see that $R_1 = f(S_1,S_2,S_3)$, so that a simple change along R_1 in the old coordinate system translates to a general three dimensional position change in our new coordinate system. The physical meaning of $\partial F/\partial R_1$ is still the projection of a position change onto R_1 and its associated change in F, but now we must treat R_1 as a general space curve in the S_i system. The partial derivative $\partial F/\partial R_1$ transforms into the S_i system according to the following rule:

$$\frac{\partial F}{\partial R_1} = \frac{\partial F}{\partial S_1}\frac{\partial S_1}{\partial R_1} + \frac{\partial F}{\partial S_2}\frac{\partial S_2}{\partial R_1} + \frac{\partial F}{\partial S_3}\frac{\partial S_3}{\partial R_1}. \quad \text{(A14)}$$

Every point in the old system maps onto a unique point in the new system, so we might expect that the new coordinate system must always have the same number of independent variables as the old one. By carefully defining the new coordinate system, however, we may be able to reduce the number of independent variables without loss of generality. For example, if we know that all position changes in the R_i coordinate frame will be restricted to a certain plane, then we could apply the special transformation by which all changes in the R_i system become changes in a two-dimensional S_i system which has its coordinate axes in this particular plane. This transformation will greatly simplify any function originally defined in the R_i system. Similarly, if R_i is a two-dimensional system, but we know that all position changes will be along a certain line, we can define this line as the only axis in our new coordinate system. By this transformation we will have reduced any function in the old system to a function with only one independent variable in the new system.

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Note added in proof

Work in progress by P. Cobbold and J. Cutler suggests that the reduced form of the compatability equations (4) may be restricted in their application. This does not apply to the standard form of the equations.